CONCERNING MEASURES IN FIRST ORDER CALCULI*

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§0. **Introduction.** The idea of treating probability as a real valued function defined on sentences is an old one (see [6] and [7], where other references can be found). Carnap's attempt to set up a theory of probability which will have a logical status analogous to that of two valued logic, is closely connected with it, cf. [1]. So far the sentences were used mainly from a "Boolean algebraic" point of view, that is, the operations that were involved were those of the sentential calculus. (The work of Carnap and his collaborators does, however, touch on probabilities which are defined for special cases of first order monadic sentences.)

A measure on a sentential calculus which assigns real values to sentences is essentially the same as a measure on the Lindenbaum-Tarski algebra of that calculus, thus its investigation falls under the study of measures on Boolean algebras. These were studied quite a lot; see [3, 5] were other references are given.

In this work the notions of a measure on a first order calculus, and of a measure-model, are introduced and investigated. This is done not from a point of view concerning the foundations of probability but with an eye to mathematical logic and measure theory; the concepts with which we shall deal form a natural generalization of the concepts of a theory and a model in the usual sense.

In §1 the notion of a measure on a first order calculus is introduced. In §2 the notion of a measure-model is defined and a theorem analogous to the completeness theorem is proved. In §3 the case of a calculus with an equality is treated. §4 is concerned with measure-models in which the measure is invariant under permutations of the individuals, and §5 contains a specific example of such a model. Whereas the propositions of §§1, 2 are analogous to similar ones concerning theories and models (in the usual sense), §§4, 5 deal with situations which are typical to measures and measure-models and have no analogous counterpart.

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* The basic definitions and concepts of this paper were first presented by the author in a contributed paper to the 1960 Congress of Logic and Methodology of Science which took place at Stanford [2]. The paper contained part of the results appearing here. Other results, unpublished yet, were obtained since then by Ryll-Nardzewski, and presented by him in a talk given at the International Symposium of Model Theory, 1963, which took place at Berkeley.

§1. **Measures.** Consider a first-order predicate calculus $\mathfrak{P}$, that is, a system consisting of individual variables, individual constants and predicate constants, as well as the sentential connectives $\sim$, $\vee$, $\wedge$, $\rightarrow$, $\equiv$ and the first order quantifiers $\exists$ and $\forall$. $\mathfrak{P}$ may or may not be with an equality. Let C be any set of individual constants, not necessarily belonging to $\mathfrak{P}$. $\mathfrak{F}(C)$ is the set of all formulas formed using the individual variables, the predicates, the connectives and quantifiers of $\mathfrak{P}$, and the individual constants of C. $\mathfrak{S}(C)$ is the set of all sentences (i.e., formulas without free variables) of $\mathfrak{F}(C)$. $\mathfrak{F}_0(C)$ is the subset of $\mathfrak{F}(C)$ consisting of all formulas in which no quantifier occurs, and $\mathfrak{S}_0(C) = \mathfrak{F}_0(C) \cap \mathfrak{S}(C)$. (Thus if $C = \emptyset$, $\mathfrak{S}_0(C) = \emptyset$). '$\vdash\phi$' means that ϕ is logically valid.

Now let C be the set of individual constants of $\mathfrak{P}$.

By a measure on $\mathfrak{P}$ we mean a function μ defined on a subset of $\mathfrak{S}(C)$ having non-negative real numbers as values, and not vanishing identically, such that the domain of μ, $D\mu$, is closed under sentential operations (i.e. if $\phi,\psi \in D\mu$ then $\sim\phi, \phi \vee \psi$ etc. belong to $D\mu$) and the following holds for all $\phi,\psi \in D\mu$:

(1) *If $\vdash\phi$ and $\vdash\psi$ then $\mu(\phi) = \mu(\psi)$*

(2) *If $\vdash \sim(\phi \wedge \psi)$ then $\mu(\phi \vee \psi) = \mu(\phi) + \mu(\psi)$*

(1) and (2) imply:

(1′) If $\phi,\psi \in D\mu$ and $\vdash\phi \equiv \psi$ then $\mu(\phi) = \mu(\psi)$.

Proof. If $\vdash\phi \equiv \psi$ then $\vdash\sim(\phi \wedge \sim\psi)$ and thus $\mu(\phi \vee \sim\psi) = \mu(\phi) + \mu(\sim\psi)$. Since $\vdash \sim(\psi \wedge \sim\psi)$ we get $\mu(\sim\psi) = \mu(\psi \vee \sim\psi) - \mu(\psi)$. Since $\vdash\phi \vee \sim\psi$ we have $\mu(\psi \vee \sim\psi) = \mu(\phi \vee \sim\psi)$ hence $\mu(\phi) - \mu(\psi) = 0$. (One can show that (2) alone or (2) together with the requirement $\mu(\phi) = 0$ whenever $\vdash\sim\phi$ do not suffice to get (1′)).

Thus, a measure can be conceived as a non-trivial, non-negative, finitely additive measure on a Boolean subalgebra of the Lindenbaum-Tarski algebra of the sentences of $\mathfrak{P}$, the subalgebra being $\{\psi/\equiv \mid \psi \in D\mu\}$, where $\psi/\equiv\; = \{\phi \mid \vdash\phi \equiv \psi\}$.

A probability on $\mathfrak{P}$ is a measure μ for which $\mu(\phi) = 1$ whenever $\phi \in D\mu$ and $\vdash\phi$.

One can consider also measures defined for formulas containing free variables as well, this however makes no essential difference, the statements and constructions which follow can be modified in an obvious way to take care of this case.

The notion of a probability is advanced here as a natural generalization to that of a theory. Whereas in the case of a theory one speaks of sentences as being true or false, in the case of a probability a sentence has a certain probability which might in general be any number between 0 and 1. A theory is a probability having only two values 0 and 1, the theory being complete if the domain of the probability is the set of all sentences. This analogy motivates the following definition of a measure-model.

§2. **Measure-models and the completeness theorem.** Let C be the set of individual constants of $\mathfrak{P}$. *A measure-model for* $\mathfrak{P}$ *is a pair* $\langle U,m\rangle$, *where* $U \supseteq C, U \neq \emptyset$, *and* m *is a measure on* $\mathfrak{S}_0(U)$. $\langle U,m\rangle$ *is a probability-model if* m *is a probability.*

A usual model consists in giving an interpretation to the predicates of the calculus as relations over a certain set U. This might be also described as a function assigning the values 0 or 1 to every expression of the form $R(a_1,\cdots,a_n)$ where R is an n-place predicate of $\mathfrak{P}$ and $a_1,\cdots,a_n \in U$. In generalizing this we consider a function having values in $[0,1]$, but this in itself is not sufficient, since, in general, the values assigned to atomic sentences do not determine unique values for sentential combinations of these sentences. Hence one has to start from a probability defined on the whole of $\mathfrak{S}_0(U)$. This, as it turns out, does determine a natural extension to $\mathfrak{S}(U)$.

THEOREM 1. *Let* $\langle U,m\rangle$ *be a measure-model, then there is a unique measure* m^* *which extends* m *to* $\mathfrak{S}(U)$ *and satisfies*:

(3) If $\phi(x) \in \mathfrak{F}(U)$ *and* x *is its only free variable then*

$$m^*(\exists x\, \phi(x)) = \sup\{m^*(\textstyle\bigvee_{i=1}^{n}\phi(a_i)) \mid a_1,\cdots,a_n \in U\}$$

('$\bigvee_{i=1}^{n}\phi(a_i)$' *stands for* '$\phi(a_1) \vee \cdots \vee \phi(a_n)$', *and the supremum is taken over all finite subsets of* U).

Proof. Since every measure can be made into a probability through multiplying by a normalizing factor it suffices to prove the theorem assuming that m is a probability. It is easily seen that (3) is equivalent to:

(3′) If $\phi(x_1,\cdots,x_k) \in \mathfrak{F}(U)$ and $x_1,\cdots,x_k$ are all its free variables then

$$m^*(\exists x_1 \cdots x_k\phi(x_1,\cdots,x_k)) = \sup\{m^*(\textstyle\bigvee_{i=1}^{n}\phi(a_{i,1},\cdots,a_{i,k})) \mid a_{1,1},\cdots,a_{n,k} \in U\}$$

("$\exists x_1 \cdots x_k$" stands for "$(\exists x_1)\cdots(\exists x_k)$")

For every $U' \subseteq U$ let $\Sigma_n(U')$ be the subset of $\mathfrak{F}(U')$ consisting of those formulas which are in prenex normal form with no more than n alternating blocks of quantifiers and, in case $n > 0$ and there are n blocks, the leftmost one consists of existential quantifiers. Let $\Pi_n(U')$ be likewise defined except that in the case of n blocks, where $n > 0$, the leftmost one is required to be a block of universal quantifiers. Now assume m_1^* and m_2^* both extend m and satisfiy (3′). If m_1^* and m_2^* coincide on all sentences in $\Pi_n(U)$ and $\psi \in \Sigma_{n+1}(U) \cap \mathfrak{S}(U)$ then $\psi = \exists x_1 \cdots x_k\phi(x_1,\cdots,x_k)$ where $\phi(x_1,\cdots,x_k) \in \Pi_n(U)$. All sentences of the form $\phi(a_{i,1},\cdots,a_{i,k})$ are in $\Pi_n(U)$ and every disjunction of sentences from $\Pi_n(U)$ is logically equivalent to a sentence of $\Pi_n(U)$, hence (3′) implies that $m_1^*(\psi) = m_2^*(\psi)$. If $\psi \in \Pi_{n+1}(U) \cap \mathfrak{S}(U)$ then $\sim\psi$ is logically equivalent to a member of $\Sigma_{n+1}(U) \cap \mathfrak{S}(U)$ hence m_1^* and m_2^* coincide on $\sim\psi$ and therefore also on ψ. It follows by induction that m_1^* and m_2^* coincide on all sentences in

$\sum_n(U)$, $\Pi_n(U)$, $n = 0,1,\cdots$. Since every sentence is logically equivalent to a sentence in prenex normal form we get $m_1^* = m_2^*$.

To prove the existence of m^* we distinguish first the case where U is countable (i.e., of power $\leqq \aleph_0$). Let $\mathfrak{M}$ be the set of all models (in the usual sense) whose universe is U (i.e., systems of the form $\langle U,\cdots\rangle$). For every $\phi \in \mathfrak{S}(U)$ let $\mathfrak{M}(\phi)$ be the set of all models in which ϕ is satisfied. $\mathfrak{M}(\phi \vee \psi) = \mathfrak{M}(\phi) \cup \mathfrak{M}(\psi)$ and $\mathfrak{M}(\sim\phi) = \mathfrak{M} - \mathfrak{M}(\phi)$, hence $\{\mathfrak{M}(\phi) \mid \phi \in \mathfrak{S}_0(U)\}$ is a Boolean algebra of subsets of $\mathfrak{M}$ and m induces on it a finitely additive measure m_1 given by: $m_1(\mathfrak{M}(\phi)) = m(\phi)$. m_1 is continuous, by which we mean that if $X_i, i = 1,2,\cdots$, is a sequence of sets of the Boolean algebra such that $X_{i+1} \subseteq X_i$, $i = 1,2,\cdots$, and $\bigcap_{i=1}^{\infty} X_i = \emptyset$ then $\lim_{i\to\infty} m_1(X_i) = 0$. If U is finite this is obvious. If U is infinite then it follows from the compactness theorem for the sentential calculus, which states that, for $\phi_i \in \mathfrak{S}_0(U)$, $i = 1,2,\cdots$, either there is an n for which $\vdash \sim \bigwedge_{i=1}^{n} \phi_i$ or $\bigcup_{i=1}^{\infty} \mathfrak{M}(\phi_i) \neq \emptyset$; consequently a decreasing sequence of sets of the Boolean algebra has an empty intersection only if some member of it is already empty. As is well-known, cf. [4, pp. 74–80], a continuous measure can be extended to a countably additive measure m_1^* on the σ-field (i.e. Boolean algebra of sets closed under countable unions and intersections) which is generated by the sets of the Boolean algebra. Since for all $\exists x\phi(x)$ in $\mathfrak{S}(U)$ we have $\mathfrak{M}(\exists x\phi(x)) = \bigcup_{a\in U} \mathfrak{M}(\phi(a))$ it follows, assuming that U is countable, that, for all $\phi \in \mathfrak{S}(U)$, $\mathfrak{M}(\phi)$ belongs to the σ-field on which m_1^* is defined. If we put $m^*(\phi) = m_1^*(\mathfrak{M}(\phi))$ then m^* extends m to $\mathfrak{S}(U)$. The countable-additivity of m_1^* implies (3).

Now let U be of any power. For every $\phi \in \mathfrak{F}(U)$ let $\phi^{\#}$ be a logically equivalent formula in prenex normal form such that whenever ϕ is in prenex normal form $\phi^{\#} = \phi$, if $\phi_1, \phi_2 \in \Pi_n(U)$ then $(\phi_1 \vee \phi_2)^{\#} \in \Pi_n(U)$ and $(\sim\phi_1)^{\#} \in \sum_n(U)$. Define m^* for sentences in prenex normal form, by induction, as follows:

$$m^*(\phi) = m(\phi) \text{ for } \phi \in \mathfrak{S}_0(U)$$

$$m^*(\exists x_1\cdots x_k\phi(x_1,\cdots,x_k)) = \sup\{m^*(\bigvee_{i=1}^{n}\phi(a_{i,1},\cdots,a_{i,k}))^{\#} \mid a_{1,1},\cdots,a_{n,k} \in U\}$$

for $\phi(x_1,\cdots,x_k) \in \Pi_n(U)$, and $m^*(\phi) = 1 - m^*((\sim\phi)^{\#})$ for $\phi \in \Pi_n(U)$.

Extend m^* to $\mathfrak{S}(U)$ by putting ,$m^*(\phi) = m^*(\phi^{\#})$. It remains to show that m^* is a measure on $\mathfrak{S}(U)$.

For every $U' \subseteq U$ let $m_{U'}$ be the restriction of m to $\mathfrak{S}_0(U')$. If U' is countable then $m_{U'}$ has an extension $m_{U'}^*$ defined on $\mathfrak{S}(U')$ and satisfying (3) with respect to U'. If T is any set of predicates of the calculus let $\mathfrak{S}_T(U')$ be the set of all the sentences of $\mathfrak{S}(U')$ all of whose predicates are in T. If $U_1 \subseteq U_2$ and $\overline{\overline{U}}_2 \leqq \aleph_0$, then U_2 is said to be *an n-elementary extension of* U_1 *with respect to* T if $m_{U_2}^*(\phi) = m_{U_1}^*(\phi)$ for all $\phi \in \sum_n(U_1)$. $\cap\, \mathfrak{S}_T(U_1)$, (from which it follows that this is also true for all $\phi \in \prod_n(U_1) \cap \mathfrak{S}_T(U_1)$). Obviously if U_3 is an n-elementary extension of U_2, and U_2 and n-elementary extension of U_1, all with respect to T, then U_3 is an n-elementary extension of U_1. Let U_i, $i = 1,2,\cdots$, be countable subsets of U such that U_{i+1} is an n-elementary extension of U_i, $i=1,2,\cdots$, all with respect to some fixed T, then, as we shall see, $U_\infty = \bigcup_{i=1}^{\infty} U_i$ is an n-elementary extension of each U_i.

First it is obvious that $m^*_{U_\infty}$ coincides with each $m^*_{U_i}$ on $\Pi_0(U_i) \cap \mathfrak{S}_T(U_i)$. Assume it coincides with each $m^*_{U_i}$ on $\Pi_j(U_i) \cap \mathfrak{S}_T(U_i)$, $j < n$. Let $\phi(x_1,\cdots,x_k) \in \Pi_j(U_i)$, assume all its predicates are in T and $x_1,\cdots,x_k$ are its free variables, for simplicity take $k = 1$.

$$m_{U_\infty}^*(\exists x\phi(x)) = \sup\{m^*_{U_\infty}(\bigvee_{t=1}^r \phi(a_t)) \mid a_1,\cdots,a_r \in U_\infty\}$$

$$m^*_{U_i}(\exists x\phi(x)) = \sup\{m^*_{U_i}(\bigvee_{t=1}^r \phi(a_t)) \mid a_1,\cdots,a_r \in U_i\}.$$

Since $\bigvee_{t=1}^r \phi(a_t)$, where $a_1,\cdots,a_r \in U_i$, is logically equivalent to a member of $\Pi_j(U_i) \cap \mathfrak{S}_T(U_i)$ and $U_i \subseteq U_\infty$ it follows that $m^*_{U_\infty}(\exists x\phi(x)) \geqq m^*_{U_i}(\exists x\phi(x))$. On the other hand, given $\varepsilon > 0$, let $a_1,\cdots,a_r$ be such that $m^*_{U_\infty}(\exists x\phi(x)) \leqq m_{U_\infty}(\bigvee_{t=1}^r \phi(a_t)) + \varepsilon$, then, for some $l \geqq i$, $a_1,\cdots,a_r \in U_l$ hence $m_{U_l}(\exists x\phi(x)) + \varepsilon \geqq m^*_{U_l}(\bigvee_{t=1}^r \phi(a_t)) + \varepsilon = m^*_{U_\infty}(\bigvee_{t=1}^r \phi(a_t)) + \varepsilon \geqq m^*_{U_\infty}(\exists x\phi(x))$. Since U_l is an n-elementary extension of U_i we have $m^*_{U_l}(\exists x\phi(x)) = m^*_{U_i}(\exists x\phi(x))$. This proves the opposite inequality. Therefore $m^*_{U_i}$ and $m^*_{U_\infty}$ coincide on $\Sigma_{j+1}(U_i) \cap \mathfrak{S}_T(U_i)$, hence also on $\Pi_{j+1}(U_i) \cap \mathfrak{S}_T(U_i)$.

Finally we claim that given any n, any countable subset U' of U, and any countable set T of predicates, there is a countable $W \supseteq U'$ such that m^*_W coincides with m^* on $\Sigma_n(W) \cap \mathfrak{S}_T(W)$. If $n = 0$ put $W = U'$. Assume it to be true for n. For every sentence $\psi = \exists x_1 \cdots x_k \phi(x_1,\cdots,x_k)$, where $\phi \in \Pi_l(U)$ for some $l \geqq 0$, there is a countable set $V(\psi)$ such that

$$m^*(\psi) = \sup\{m^*(\bigvee_{t=1}^r \phi(a_{t,1},\cdots,a_{t,k}))^\# \mid a_{1,1},\cdots,a_{r,k} \in V(\psi)\}$$

If ψ is not of this form put $V(\psi) = \emptyset$. We can also assume that if ψ' is obtained from ψ by a change of bound variables then $V(\psi) = V(\psi')$. Put $U_1 = U'$ and let W_1 be the countable subset including U_1 such that $m^*_{W_1}$ coincides with m^* on $\Sigma_n(W_1) \cap \mathfrak{S}_T(W_1)$. Put $U_2 = W_1 \cup \bigcup_{\psi \in \mathfrak{S}_T(W_1)} V(\psi)$. Since T is countable so is U_2. Extend U_2 to W_2 in the same way that U_1 was extended to W_1, then extend W_2 to U_3 and so on. We get a sequence $U_1, W_1, \cdots, U_i, W_i, \cdots$ in which $U_i \subseteq W_i \subseteq U_{i+1}$, $i = 1, 2, \cdots$ and $m^*_{W_i}$ coincides with m^* on $\Sigma_n(W_i) \cap \mathfrak{S}_T(W_i)$. Therefore, every W_{i+1} is an n-elementary extension of W_i with respect to T. Put $W = \bigcup_{i=1}^{\infty} W_i$, then W is an n-elementary extension of each W_i, therefore m^*_W coincides with m^* on $\Sigma_n(W_i) \cap \mathfrak{S}_T(W_i)$ for all i, hence they coincide on $\Sigma_n(W) \cap \mathfrak{S}_T(W)$. If $\psi = \exists x_1 \cdots x_k \phi(x_1,\cdots,x_k)$, where $\phi \in \Pi_n(W)$ and all its predicates are in T, then

$$m^*_W(\psi) = \sup\{m^*_W(\bigvee_{t=1}^r \phi(a_{t,1},\cdots,a_{t,k})) \mid a_{1,1},\cdots,a_{r,k} \in W\}$$

since m^*_W and m^* coincide on $\Pi_n(W) \cap \mathfrak{S}_T(W)$ it follows that $m^*(\psi)$ is defined by taking a supremum over a bigger set hence $m^*(\psi) \geqq m^*_W(\psi)$. On the other hand, for some i, $\psi \in \mathfrak{S}_T(W_i)$ hence $V(\psi) \subseteq W$ from which it follows that $m^*(\psi) \leqq m^*_W(\psi)$, hence equality holds.

Now given any two sentences of $\mathfrak{S}(U)$, ϕ, ψ, there are countable T and W and a number n such that $\phi^{\#}, \psi^{\#}, (\phi \vee \psi)^{\#} \in \Sigma_n(W) \cap \mathfrak{S}_T(W)$, and since every m_W^* is a measure this implies that m^* is a measure as well. q.e.d.

Note that (3) is equivalent to

$$(3'') \quad m^*(\forall x\phi(x)) = \inf\{m^*(\textstyle\bigwedge_{i=1}^n \phi(a_i)) \mid a_1, \cdots, a_n \in U\}, \text{ for all sentences } \forall x\phi(x).$$

The inductive definition of m^* which is based on (3′) and used in the proof for the case in which U is uncountable, can be used to prove the theorem directly for both countable and uncountable U's; one has to show that m^* is a measure, and this can be done by defining the transformation $\phi \to \phi^{\#}$ in some particular suitable way, and using for the case where U is infinite some versions of Herbrand's theorem.

Note that the notion of an elementary submodel can be carried over to the case of measure-models as indicated in the proof of the theorem.

Thus we get:

A measure-model $\langle U_1, m\rangle$ is a submodel of $\langle U, m\rangle$ if $U_1 \subseteq U$ and m_1 is m restricted to $\mathfrak{S}_0(U_1)$. It is an elementary submodel if it is a submodel and, for all $\phi \in \mathfrak{S}(U_1)$, $m^*(\phi) = m_1^*(\phi)$, where m^* and m_1^* are the unique extensions of m and m_1 satisfying (3) *for U and U_1, respectively.*

(If $\langle U_1, m_1\rangle$ is to be a measure-model for the same calculus an additional stipulation should be made, to the effect that all individual constants of the calculus belong to U_1.)

The techniques involving elementary measure-submodels, of which some were used in the proof, are fully analogous to the techniques used for ordinary models, in particular:

If $\langle U, m\rangle$ is a measure-model and $U' \subseteq U$ then U' can be extended to U'' so that, if m'' is m restricted to U'', then $\langle U'', m''\rangle$ is an elementary submodel of $\langle U, m\rangle$. U'' can be chosen to be of power not exceeding the maximum of $\aleph_0$, $\overline{\overline{U'}}$, and the power of the family of predicates of the calculus.

As was pointed out in the proof of Theorem 1, every measure-model $\langle U, m\rangle$ induces a measure on the Boolean algebra $\{M(\phi) \mid \phi \in \mathfrak{S}_0(U)\}$, where $M(\phi)$ is the set of all ordinary models with domain U in which ϕ is satisfied. This measure is continuous and can be extended to a countably additive measure on the σ-field generated by this Boolean algebra. In case $\overline{\overline{U}} \leq \aleph_0$ this σ-field contains all sets $M(\phi)$ where $\phi \in \mathfrak{S}(U)$, and $m^*(\phi)$ is equal to the value of the extension for $M(\phi)$. If $\overline{\overline{U}} > \aleph_0$ and we define m' by: $m'(M(\phi)) = m^*(\phi)$, $\phi \in \mathfrak{S}(U)$, then the compactness theorem implies the continuity of m'. (If $\bigcap_{i=1}^n M(\phi_i) \neq \emptyset$ for all n, then, since every ϕ_i involves only finitely many predicates and members of U, there is a model with domain U in which all are satisfied, hence $\bigcap_{i=1}^{\infty} M(\phi_i) \neq \emptyset$.). Therefore, in any case, m' defined as above is a continuous measure on $\{M(\phi) \mid \phi \in \mathfrak{S}(U)\}$ and as such can be extended to a countably additive measure on the σ-field generated by $\{M(\phi) \mid \phi \in \mathfrak{S}(U)\}$. On the other hand, if m' is a countably additive measure on the σ-field generated by $\{M(\phi) \mid \phi \in \mathfrak{S}(U)\}$ and U is countable, then m'', defined by: $m''(\phi) = m'(M(\phi))$, $\phi \in \mathfrak{S}(U)$, satisfies (3). This is no longer true if $\bar{\bar{U}} > \aleph_0$, since in that case every finitely additive measure on $\{M(\phi) \mid \phi \in \mathfrak{S}(U)\}$ is continuous and can be extended to a countably additive measure on the σ-field, while one can easily construct measures on $\mathfrak{S}(U)$ which do not satisfy (3).

A measure-model $\langle U, m \rangle$ *is said to determine the measure* μ *(on* $\mathfrak{P}$*), if* μ *is the restriction of* m^* *to* $D\mu$*, where* m^* *is the unique extension of* m *satisfying* (3). $\langle U, m \rangle$ *is said to be a model of* μ *if it determines* μ.

The following analogy to the completeness theorem holds.

THEOREM 2. *Every measure* μ *on* $\mathfrak{P}$ *has a measure-model whose power is* $\aleph_0$ + *the power of the set of all sentences of* $\mathfrak{P}$.

The proof of this theorem is analogous to the proof of the completeness theorem which uses the prime ideal theorem to extend an ideal in the Lindenbaum-Tarski algebra to a prime ideal. Here we extend a measure on a subalgebra to a measure on the whole algebra. We quote the following result [8, pp. 268–270].

(4) Assume $\mathfrak{B}'$ is a Boolean subalgebra of $\mathfrak{B}$ and m is a measure on $\mathfrak{B}'$. For every $b \in \mathfrak{B}$ put $m^+(b) = \inf\{m(b') \mid b' \in \mathfrak{B}' \text{ and } b' \geqq b\}$, $m^-(b) = \sup\{m(b') \mid b' \in \mathfrak{B}' \text{ and } b' \leqq b\}$. Then, given any element $b \in \mathfrak{B}$ and any $0 \leqq \theta \leqq 1$, the equation

$$m'(b \cdot b' + \bar{b} \cdot b'') = \theta m^+(b \cdot b') + (1 - \theta) m^-(\bar{b} \cdot b''), \quad b', b'' \in \mathfrak{B}'$$

defines a measure m' which extends m to the subalgebra generated by $\mathfrak{B}'$ and b. ("+", "·" and "$^-$" denote here the join, meet and complement operations in the Boolean algebra.)

This together with the axiom of choice implies that every measure on a Boolean subalgebra can be extended to the whole algebra.

Proof of Theorem 2. Let C_0 be the set of individual constants of $\mathfrak{P}$. For every sentence ϕ of $\mathfrak{S}(C_0)$ of the form $\exists x \psi(x)$ let a_ϕ be a new individual constant, $a_{\phi_1} \neq a_{\phi_2}$ if $\phi_1 \neq \phi_2$. Let C_1 be the set of all a_ϕ thus obtained; in general let C_{i+1} be the set of all individual constants a_ϕ, where $\phi \in \mathfrak{S}(\bigcup_{j=1}^{i} C_j) - \mathfrak{S}(\bigcup_{j=1}^{i-1} C_j)$ and is of the form $\exists x \psi(x)$. Put $U = \bigcup_{i=0}^{\infty} C_i$.

Let $\mathfrak{S}'$ be the set of all sentences in $\mathfrak{S}(U)$ of the form $\exists x \psi(x) \rightarrow \psi(a_\phi)$ where $\phi = \exists x \psi(x)$.

For all $\sigma \in \mathfrak{S}(U)$ let $\sigma/\equiv$ be the element of the Lindenbaum-Tarski algebra represented by σ. Let $\mathfrak{B}$ be the whole algebra $\{\sigma/\equiv \mid \sigma \in \mathfrak{S}(U)\}$ and $\mathfrak{B}'$ the subalgebra generated by $\{\sigma/\equiv \mid \sigma \in D\mu \cup \mathfrak{S}'\}$. Every element of $\mathfrak{B}'$ is of the form $(\sigma_1 \wedge \psi_1) \vee \cdots \vee (\sigma_n \wedge \psi_n)/\equiv$ where $\sigma_i \in D\mu$, $i = 1, \cdots, n$ and ψ_i is of the form $\bigwedge_j \psi_{i,j}$ where, for every j, $\psi_{i,j} \in \mathfrak{S}'$ or $\psi_{i,j}$ is a negation of a member of $\mathfrak{S}'$. Let $\mathfrak{J}$ be the ideal in $\mathfrak{B}$ generated by all elements of the form $\sim \phi/\equiv$ where $\phi \in \mathfrak{S}'$. Assume that $\sigma \in \mathfrak{S}(C_0)$ and $\sigma/\equiv \in \mathfrak{J}$. There are τ_i's which are negations of members of $\mathfrak{S}'$ so that $\vdash \sigma \to \bigvee_{i=1}^{n} \tau_i$. Let $\tau_i = \sim(\exists x \psi_i(x) \to \psi(a_{\phi_i}))$, where $\phi_i = \exists x \psi_i(x)$, and we can assume that $\phi_i \neq \phi_j$ for $i \neq j$. Let $a_{\phi_i} \in C_{k_i}$; assume, with no loss of generality, that $k_n \geqq k_i$ for all $i \leqq n$. It follows that a_{ϕ_n} does not occur in σ as well as in any τ_i where $i \neq n$, hence, using well-known rules of first order logic, we get $\vdash \sigma \to (\exists x \psi_n(x) \wedge \forall y \sim \psi_n(y)) \vee \bigvee_{i=1}^{n-1} \tau_i$. If $n = 1$ we get $\vdash \sim \sigma$, otherwise $\vdash \sigma \to \bigvee_{u=1}^{n-1} \tau_i$ and by continuing the argument we get $\vdash \sim \sigma$. This implies that, for all $\sigma_1, \sigma_2 \in \mathfrak{S}(C_0)$, $(\sigma_1/\equiv)/\mathfrak{J} = (\sigma_2/\equiv)/\mathfrak{J}$ iff $\vdash \sigma_1 \equiv \sigma_2$. For every $\phi/\equiv \in \mathfrak{B}'$ there is $\psi \in D\mu$ such that $(\phi/\equiv)/\mathfrak{J} = (\psi/\equiv)/\mathfrak{J}$, consequently one can define a measure μ' on $\mathfrak{B}'/\mathfrak{J}$ by putting $\mu'(\phi/\equiv)/\mathfrak{J}) = \mu(\psi)$, where $\psi \in D\mu$ is such that $(\phi/\equiv)/\mathfrak{J} = (\psi/\equiv)/\mathfrak{J}$. Now extend μ' to a measure μ^* on $\mathfrak{B}/\mathfrak{J}$. Define m^* by $m^*(\psi) = \mu^*((\psi/\equiv)/\mathfrak{J})$, for all $\psi \in \mathfrak{S}(U)$. m^* is a measure on $\mathfrak{S}(U)$ for which $m^*(\exists x \psi(x) \to \psi(a_\phi)) = 1$ whenever $\phi = \exists x \psi(x)$. Consequently $m^*(\exists x \psi(x)) \leqq m^*(\psi(a_\phi))$, since the opposite inequality holds trivially one gets $m^*(\exists x \psi(x)) = m^*(\psi(a_\phi))$, from which it follows that m^* satisfies (3). m^* coincides with μ on $D\mu$, hence if m is the restriction of m^* to $\mathfrak{S}_0(U)$, then $\langle U, m \rangle$ is a measure-model for μ. q.e.d.

Note that the measure-model constructed in this proof has the additional property that for every sentence $\exists x \psi(x)$ in $\mathfrak{S}(U)$ there is a member $a \in U$ such that $m^*(\exists x \psi(x)) = m^*(\psi(a))$.

§3. **Strict equality**. Let $\langle U, m \rangle$ be a measure-model for $\mathfrak{P}$ and assume that $\mathfrak{P}$ has an equality, $=$. *$\langle U, m \rangle$ is a model with strict equality if $m(a = a') = 0$ whenever $a, a' \in U$ and $a \neq a'$.*

THEOREM 3. *Let $\mathfrak{P}$ be a first-order calculus with equality, and let C be the set of individual constants of $\mathfrak{P}$. If μ is a probability defined for all sentences of $\mathfrak{P}$, then a necessary and sufficient condition for μ to have a probability-model with strict equality is:*

(5) *For all $a, a' \in C$, $a \neq a'$ implies $\mu(a = a') = 0$.*

(6) *For every k, $\mu(\exists x_1 \cdots x_k \forall y (\bigvee_{i=1}^{k} y = x_i))$ is either 0 or 1.*

We formulate the theorem in terms of probability rather then measure only for the sake of convenience. The theorem remains true if 'probability' is replaced, throughout, by 'measure' provided that, in (6) '1' is replaced by $\mu(\phi \vee \sim \phi)$, where $\phi \in D\mu$.

Proof. The necessity of (5) is obvious. Since $\vdash \bigwedge_{i<j} \bigwedge_{j \leqq k+1} (a_i \neq a_j) \to \sim \exists x_1 \cdots x_k \forall y (\bigvee_{i=1}^{k} y = x_i)$ (where "$a_i \neq a_j$" stands for "$\sim a_i = a_j$"), it follows that if $\langle U, m \rangle$ is a probability-model with strict equality then $m^*(\sim \exists x_1 \cdots x_k \forall y (\bigvee_{i=1}^{k} y = x_i)) = 1$ whenever m^* is an extension of m and $k < \bar{\bar{U}}$. If $k \geqq \bar{\bar{U}}$, say $U = \{a_1, \cdots, a_j\}$, $k \geqq j$, then $\vdash \bigvee_{i=1}^{j} a = a_i$ for all $a \in U$, hence if m^* is the unique extension of m satisfying (3), $m^*(\forall y (\bigvee_{i=1}^{j} y = a_i)) = 1$, which implies $m^* (\exists x_1 \cdots x_2 \forall y (\bigvee_{i=1}^{k} y = x_i)) = 1$. Therefore (6) is necessary.

The sufficiency of (5) and (6) follows from:

(7) Let U be any set of individual constants and m a probability on $\mathfrak{S}(U)$ satisfying (6), such that for all $a_1, a_2 \in U$, $a_1 \neq a_2$ implies $m(a_1 = a_2) = 0$. Then either $m(\exists x(\psi x)) = \sup\{m(\bigvee_{i=1}^{n}\psi(a_i)) \mid a_1, \cdots, a_n \in U\}$ for all sentences $\exists x\psi(x)$ in $\mathfrak{S}(U)$, or else, given any particular sentence $\exists x\psi(x)$ one can add a new individual constant, a', to U and extend m to a probability m' on $\mathfrak{S}(U')$, $U' = U \cup \{a'\}$, so that, for all $a \in U$, $m'(a = a') = 0$ and $m'(\exists x\psi(x)) = \sup\{m'(\bigvee_{i=1}^{n}\psi(a_i)) \mid a_1, \cdots, a_n \in U'\}$.

Once (7) is proved the proof of the existence of a probability-model with strict equality follows the same lines as the proof of Theorem 2. Namely, putting $C_0 = C$ we define C_i as before, and we keep adding step by step the new constants a_ϕ and extending the probability according to (7), so as to make it satisfy (3) for the particular sentence ϕ. This is done for every C_i by transfinite induction. Either we get at some stage a probability already satisfying (3) or else the process can be carried on and we get such a probability on $\bigcup_{i=0}^{\infty} C_i$. The probability-model thus constructed determines μ and is with strict equality.

Proof of (7): If, for some k, $m(\exists x_1 \cdots x_k \forall y(\bigvee_{i=1}^{k} y = x_i)) = 1$ let k_0 be the smallest k of this property. Then (since $m(a_1 = a_2) = 0$ whenever $a_1 \neq a_2$) $\overline{\overline{U}} \leqq k_0$. If $\overline{\overline{U}} = k_0$, say $U = \{a_1, \cdots, a_{k_0}\}$, we get

$$\vdash \bigwedge_{i<j} \bigwedge_{j=1}^{k_0} (a_i \neq a_j) \wedge \exists x_1 \cdots x_{k_0} \forall y(\bigvee_{i=1}^{k_0} y = x_i) \rightarrow [\exists x\psi(x) \equiv \bigvee_{i=1}^{k_0} \psi(a_i)]$$

which implies that m satisfies (3) with respect to U. Thus we can assume that either $\overline{\overline{U}} < k_0$ or, for every k, $m(\exists x_1 \cdots x_k \forall y(\bigvee_{i=1}^{k} y = x_i)) = 0$; each of these implies that $m(\forall y(\bigvee_{i=1}^{n} y = a_i)) = 0$ whenever $a_1, \cdots, a_n \in U$. Assume $a' \notin U$, $U' = U \cup \{a'\}$, and let $\mathfrak{S}_1$ be the set of all sentential combinations of sentences from $\mathfrak{S}(U)$ and sentences of the form $a' = a$, $a \in U$. Let $\mathfrak{B}$, $\mathfrak{B}_1$, $\mathfrak{B}'$ be the Lindenbaum-Tarski algebras of $\mathfrak{S}(U)$, $\mathfrak{S}_1$, $\mathfrak{S}(U')$, respectively, and $\mathfrak{J}$ the ideal of $\mathfrak{B}_1$ generated by $\{a' = a / \equiv \mid a \in U\}$. For every $\phi \in \mathfrak{S}_1$ there is a $\psi \in \mathfrak{S}(U)$ such that $(\phi / \equiv)/\mathfrak{J} = (\psi / \equiv)/\mathfrak{J}$. Assume that $\phi \in \mathfrak{S}(U)$ and $\phi / \equiv \in \mathfrak{J}$, then $\vdash \phi \rightarrow \bigvee_{i=1}^{n} a' = a_i$ for some $a_1, \cdots, a_n$ in U. Since a' does not occur in ϕ we get $\vdash \phi \rightarrow \forall y(\bigvee_{i=1}^{n} y = a_i)$, hence $m(\phi) = 0$. It follows that one can define a measure $\hat{m}$ on $\mathfrak{B}_1/\mathfrak{J}$ by putting $\hat{m}((\phi / \equiv)/\mathfrak{J}) = m(\psi)$, where $\psi \in \mathfrak{S}(U)$ is such that $(\phi / \equiv)/\mathfrak{J} = (\psi / \equiv)/\mathfrak{J}$. Now put $m_0(\psi) = \hat{m}((\psi / \equiv)/\mathfrak{J})$, then m_0 extends m to $\mathfrak{S}_1$ and $m_0(a' = a) = 0$ for all $a \in U$.

Given a sentence of $\mathfrak{S}(U)$ of the form $\exists x\psi(x)$ put $\xi = m(\exists x\psi(x))$, $\eta = \sup\{m(\bigvee_{i=1}^{k}\psi(b_i)) \mid b_1, \cdots, b_k \in U\}$ and let $a_1, a_2, \cdots, a_n, \cdots$ be a sequence of members of U such that $\eta = \lim_{n \to \infty} m(\bigvee_{i=1}^{n}\psi(a_i))$. Put $\sigma_n = \psi(a') \wedge \bigwedge_{i=1}^{n} \sim\psi(a_i)$. We claim the following statement:

(8) If $\mathfrak{B}'$ is a Boolean algebra, $\mathfrak{B}_1$ a subalgebra, m_0 a measure on $\mathfrak{B}_1$, and $\gamma_1, \gamma_2, \cdots$ a sequence of members of $\mathfrak{B}'$ such that $\gamma_i \geqq \gamma_{i+1}$, $i = 1, 2, \cdots$, then m_0

can be extended to a measure m' on $\mathfrak{B}'$ so that $m'(\gamma_i) = m_0^+(\gamma_i)$ for all γ_i (m_0^+ is defined in (4)).

To prove it we use (4). Let $\mathfrak{B}^i$ be the subalgebra generated by $\mathfrak{B}_1$ and $\{\gamma_1, \cdots, \gamma_i\}$. First extend m_0 to a measure m_1 on B^1 so that $m_1(\delta \cdot \gamma_1) = m_0^+(\delta \cdot \gamma_1)$ for all $\delta \in \mathfrak{B}_1$; continue this process, extending m_i to a measure m_{i+1} on $\mathfrak{B}^{i+1}$ so that $m_{i+1}(\delta \cdot \gamma_{i+1}) = m_i^+(\delta \cdot \gamma_{i+1})$ for all $\delta \in \mathfrak{B}^i$. Finally let m_∞ be the measure defined on $\bigcup_{i=1}^\infty \mathfrak{B}^i$ which coincides with each m_i on $\mathfrak{B}^i$, and take m' to be any extension of m_∞ to $\mathfrak{B}'$. All that is required to show is that $m_{i-1}^+(\gamma_i) = m_0^+(\gamma_i)$, $i = 1, 2, \cdots$. It suffices to show that $m_j^+(\gamma_i) = m_{j-1}^+(\gamma_i)$ for all $j < i$. $m_1^+(\gamma_i) = \inf\{m_1(\delta) \mid \delta \in \mathfrak{B}^1 \,\&\, \delta \geqq \gamma_i\}$. If $\delta \in \mathfrak{B}^1$ then $\delta = \delta' \cdot \gamma_1 + \delta'' \cdot \bar{\gamma}_1$ where $\delta', \delta'' \in \mathfrak{B}_1$. Since $\gamma_1 \geqq \gamma_i$ we get: $\delta \geqq \gamma_i$ iff $\delta' \geqq \gamma_i$, hence $m_1^+(\gamma_i) = \inf\{m_1(\delta \cdot \gamma_1) \mid \delta \in \mathfrak{B}_1 \,\&\, \delta \geqq \gamma_i\}$. For $\delta \in \mathfrak{B}_1$, $m_1(\delta \cdot \gamma_1) = m_0^+(\delta \cdot \gamma_1) = \inf\{m_0(\delta') \mid \delta' \in \mathfrak{B}_1 \,\&\, \delta' \geqq \delta \cdot \gamma\}$ therefore $m_1^+(\gamma_i) = \inf\{\inf\{m_0(\delta') \mid \delta' \in \mathfrak{B}_1 \,\&\, \delta' \geqq \delta \cdot \gamma_1\} \mid \delta \in \mathfrak{B}_1 \,\&\, \delta \geqq \gamma_i\}$ $= \inf\{m_0(\delta') \mid \delta', \delta \in \mathfrak{B}_1 \,\&\, \delta' \geqq \delta \cdot \gamma_1 \,\&\, \delta \geqq \gamma_i\}$, where the last infimum is taken over all δ, δ' satisfying the conditions. But since $\gamma_1 \geqq \gamma_i$ this last expression is actually $\inf\{m_0(\delta') \mid \delta' \in \mathfrak{B}_1 \,\&\, \delta' \geqq \gamma_i\}$, that is, $m_0^+(\gamma_i)$, hence $m_1^+(\gamma_i) = m_0^+(\gamma_i)$; the proof for any $j < i$ is the same. (A somewhat less obvious argument proves also the statement obtained, from (8), by requiring $\gamma_i \leqq \gamma_{i+1}$ instead of $\gamma_i \geqq \gamma_{i+1}$, $i = 1, 2, \cdots$.)

Using (8) where $\gamma_n = \sigma_n / \equiv$, $n = 1, 2, \cdots$ (obviously $\vdash \sigma_{n+1} \to \sigma_n$) one derives the existence of a probability m' on $\mathfrak{S}(U')$ which extends m_0 and satisfies, for every n, $m'(\sigma_n) = \inf\{m_0(\phi) \mid \phi \in \mathfrak{S}_1 \,\&\, \vdash \sigma_n \to \phi\}$. Assume $\vdash \sigma_n \to \phi$ where $\phi \in \mathfrak{S}_1$. ϕ is logically equivalent to a sentence of the form $\bigvee_{i \in I} \phi_i \wedge \tau_i(a')$, where $\phi_i \in \mathfrak{S}(U)$ and $\tau_i(a')$ is a conjunction of sentences of the form $a' = b$ and negations of such sentences, $b \in U$. Let I_1 be the subset of I consisting of all i's for which all the conjuncts of $\tau_i(a')$ are of the form $a' \neq b$, and let $b_1, \cdots, b_k$ be all the members of U which appear in some conjunct of some $\tau_i(a')$, of the form $a' = b_i$. Then $\vdash \sigma_n \wedge \bigwedge_{i=1}^k a' \neq b_i \to \bigvee_{i \in I_1} (\phi_i \wedge \tau_i(a'))$. Replacing a' by x and taking existential generalization we get (since a' does not occur in ψ or in ϕ_i) $\vdash (\exists x(\psi(x) \wedge \bigwedge_{i=1}^k x \neq b_i)) \wedge \bigwedge_{i=1}^n \sim \psi(a_i) \to \bigvee_{i \in I_1} (\phi_i \wedge \exists x \tau_i(x))$ therefore $\vdash \exists x \psi(x) \wedge \bigwedge_{i=1}^k \sim \psi(b_i) \wedge \bigwedge_{i=1}^n \sim \psi(a_i) \to \bigvee_{i \in I_1} (\phi_i \wedge \exists x \tau_i(x))$. $m_0(\tau_i(a')) = 1$ for $i \in I_1$ hence, the value of m_0 for the right side of the conditional is $m_0(\bigvee_{i \in I_1} \phi_i)$, which implies $m_0(\bigvee_{i \in I_1} \phi_i) \geqq m_0(\exists x \psi(x)) - m_0(\bigvee_{i=1}^k \psi(b_i) \vee \bigvee_{i=1}^n \psi(a_i)) \geqq \xi - \eta$ hence $m_0(\phi) \geqq m_0(\bigvee_{i \in I} \phi_i) \geqq \xi - \eta$. Consequently $m'(\sigma_n) \geqq \xi - \eta$, which yields $m'(\psi(a') \vee \bigvee_{i=1}^n \psi(a_i)) \geqq m_0(\bigvee_{i=1}^n \psi(a_i)) + \xi - \eta$. Letting $n \to \infty$ one gets $\lim_{n \to \infty} m'(\psi(a') \vee \bigvee_{i=1}^n \psi(a_i)) \geqq \xi$. This concludes the proof.

Note that, unlike the proof of Theorem 2, the construction used in this proof does not guarantee that for every $\exists x \psi(x)$ there is a member a of U for which $m^*(\exists x \psi(x)) = m^*(\psi(a))$. This is no accident. Given any n, one can construct a measure μ on a first-order calculus which consists of an equality and finitely many monadic predicates, so that μ has measure-models with strict equality, but for every measure-model $\langle U, m \rangle$, of μ, which is with strict equality, there is at least one predicate P, of the calculus, such that $\mu(\exists xP(x)) > m(\bigvee_{i=1}^{n-1}P(a_i))$, for all $a_1, \cdots, a_{n-1} \in U$. We give the construction without the proof. The monadic predicates are $P_1, \cdots, P_{n(n-1)^2+1}$ and Q_s, where S ranges over all subsets of $\{1, 2, \cdots, n(n-1)^2+1\}$ whose power is n. μ is any measure having measure-models with strict equality and satisfying the following: $\mu(\exists xP_i(x)) = 1$ and $\mu(\exists x(P_i(x) \wedge P_j(x))) = 0$, for all $i \neq j$, $\mu(\forall x, y(Q_s(x) \wedge Q_s(y) \rightarrow x = y)) = 1$, for all S, and, $\mu(\forall x(P_i(x) \rightarrow Q_s(x))) = 1/n$ whenever $i \in S$. One has, of course, to show that such measures exist and the easiest way of doing it is to construct a measure-model with strict equality which determines a measure satisfying these equations.

On the other hand if $\mathfrak{P}$ has countably many sentences and finitely many individual constants, and μ has measure-models with strict equality, then μ has also a measure-model $\langle U, m\rangle$, with strict equality, such that, for all $\exists x\phi(x)$ in $\mathfrak{S}(U)$, there are $a_1, \cdots, a_k$ in U for which $m^*(\exists x\phi(x)) = m^*(\bigvee_{i=1}^{k}\phi(a_i))$. This follows from a slight modification of the proof of Theorem 3. Consider the sets C_i, $i = 1, 2, \cdots$, since $\mathfrak{P}$ has countably many sentences each of them would be countable, hence one can arrange $\bigcup_{i=1}^{\infty} C_i$ in a sequence. One proceeds now to extend the measure by adding at each step the first a_ϕ (in the sequence) for which all the constants a_ψ which occur in ϕ were added before. Since by adding $a_{\exists x\phi(x)}$ the measure is made to satisfy (3) with respect to $\exists x\phi(x)$, and since after each addition we still get a measure whose domains are all the sentences on a finite set of individual constants (the set of individual constants of $\mathfrak{P}$ is assumed to be finite), it follows that the measure-model one gets has the required property.

The above mentioned assertion is not true if $\mathfrak{P}$ has infinitely many individual constants. Let P be a monadic predicate of $\mathfrak{P}$ and consider a measure μ such that $\mu(\forall x, y(P(x) \wedge P(y) \rightarrow x = y)) = 1$ and $\mu(P(a_i)) = \varepsilon_i$ where a_i, $i = 1, 2, \cdots$, are individual constants of $\mathfrak{P}$, $\varepsilon_i > 0$ for all i, and $\sum_i \varepsilon_i = 1$. It is easily seen that μ has measure-models with strict equality, but in every model $\langle U, m\rangle$ of this kind $m^*(\exists xP(x)) > m(\bigvee_{i=1}^{k}P(b_i))$ for all $b_1, \cdots, b_k \in U$. The assertion is also not true if $\mathfrak{P}$ has no individual constants but uncountably many predicates. Let $\{P_\lambda, Q_\lambda\}_{\lambda < \omega_1}$ be the family of monadic predicates of $\mathfrak{P}$, where ω_1 is the first uncountable ordinal. For every $0 < \lambda < \omega_1$ let $\{\varepsilon_{\lambda,\nu}\}_{\nu<\lambda}$ be a set of real numbers such that $\varepsilon_{\lambda,\nu} > 0$ for all $\nu < \lambda$ and $\sum_{\nu<\lambda}\varepsilon_{\lambda,\nu} = 1$. One can show that there are measure models with strict equality determining a measure μ such that: $\mu(\exists xP_\lambda(x)) = 1$ for all $\lambda < \omega_1$, $\mu(\exists x(P_\lambda(x) \wedge P_\nu(x)) = 0$ whenever $\lambda \neq \nu$, $\mu(\forall x, y(Q_\lambda(x) \wedge Q_\lambda(y) \rightarrow x = y)) = 1$ for all $\lambda < \omega_1$, and $\mu(\forall x(P_\nu(x) \rightarrow Q_\lambda(x))) = \varepsilon_{\lambda,\nu}$ for all $0 \leqq \nu < \lambda < \omega_1$. One can also show that if $\langle U, m\rangle$ is such a model then there are ω_1 many λ's such that $m^*(\exists xQ_\lambda(x)) > m(\bigvee_{i=1}^{k}Q_\lambda(b_i))$ for all $b_1, \cdots, b_k \in U$.

As is easily seen, a probability μ on $\mathfrak{P}$ (not necessarily defined on all the sentences) has a probability-model with strict equality iff some extension of it to the whole set of sentences of $\mathfrak{P}$ has such a model.

A necessary and sufficient condition for the existence of an extension satisfying (5) is:

(9) If $\phi \in D\mu$ and $\vdash \phi \to \bigvee_{i=1}^{n} a_i = b_i$, where $a_i, b_i \in C$ and $a_i \neq b_i$ for $1 \leqq i \leqq n$, then $\mu(\phi) = 0$.

Put $\tau_k = \exists x_1 \cdots x_k \forall y (\bigvee_{i=1}^{k} y = x_i)$. A necessary and sufficient condition for the existence of an extension satisfying (6) is:

(10) If $\phi_1, \phi_2 \in D\mu$ and, for some k, $\vdash \phi_1 \to \tau_k$ and $\vdash \tau_k \to \phi_2$, then $\mu(\phi_1) > 0$ implies $\mu(\phi_2) = 1$.

A necessary and sufficient condition for the existence of an extension satisfying (5) and (6) is:

(11) For all $\phi_1, \phi_2 \in D\mu$ and all k, if $\vdash \bigwedge_{i=1}^{n}(a_i \neq b_i) \wedge \phi_1 \to \tau_k$ and $\vdash \tau_k \to \bigvee_{i=1}^{n}(a_i = b_i) \vee \phi_2$, where $a_i, b_i \in C$ and $a_i \neq b_i$ for $1 \leqq i \leqq n$, then $\mu(\phi_1) > 0$ implies $\mu(\phi_2) = 1$.

The proofs of these statements use (4) and techniques similar to those employed hitherto; they are omitted here.

All this holds for measures provided that "1" is replaced by "$\mu(\phi \vee \sim \phi)$".

§4. **Symmetric measure-models.** *A measure-model $\langle U, m \rangle$ is said to be symmetric in U', where $U' \subseteq U$, if for every sentence $\phi(a_1, \cdots, a_n)$ of $\mathfrak{S}(U)$, in which $a_1, \cdots, a_n$ are all the occurring members of U', $m(\phi(a_1, \cdots, a_n)) = m(\phi(\pi(a_1), \cdots, \pi(a_n)))$ whenever π is a permutation of U'.*

THEOREM 3. *If $\mathfrak{P}$ has a countable set of sentences and μ is any measure on $\mathfrak{P}$ then μ has a measure-model $\langle U, m \rangle$ which is symmetric in $U - C$, where C = set of individual constants of $\mathfrak{P}$.*

Proof. Let U be any countable set which includes C, for which $A = U - C \neq \emptyset$. Let F be the set of all functions mapping A into A and let F_0 be the set of all functions whose domain and range are finite subsets of A. If $f \in F$ and $A' \subseteq A$ then $f | A'$ is the restriction of f to A'. For every $g \in F_0$ put $[g] = \{f | f \in F$ and $f | Dg = g\}$ and let F^* be the σ-field generated by $\{[g] | g \in F_0\}$. Let ν be a countably-additive measure on F^* satisfying:

(i) $\nu(F) = 1$.

(ii) For every $g \in F_0$ if h is a one to one mapping of a finite subset of A onto Dg then $\nu([g \circ h]) = \nu([g])$, where $g \circ h$ is g composed with h, $g \circ h(x) = g(h(x))$. (That is to say, $\nu([g])$, depends only on the sequence of the values of g.).

(iii) For every $a \in A$ $\nu(\{f | f \in F \,\&\, a \in \mathfrak{A}f\}) = 1$, where $\mathfrak{A}f$ = range of f (the countability of A implies that $\{f | a \in \mathfrak{A}f\} \in F^*$).

A measure ν satisfying (i)–(iii) is easily arrived at. Say $A = \{a_1, a_2, \cdots\}, a_i \neq a_j$ if $i \neq j$ (the sequence being infinite or finite). By identifying every f in F with the sequence $f(a_1), f(a_2), \cdots$, F is identified with a cartesian power of A, the number of coordinates being equal to $\overline{\overline{A}}$. Let ν' be the countably-additive measure on all subsets of A obtained by putting $v'(\{a_i\}) = \varepsilon_i$ where every $\varepsilon_i > 0$ and $\sum_i \varepsilon_i = 1$. If we now take v to be the product measure on F then v satisfies (i), (ii), (iii).

Now let m be any measure on $\mathfrak{S}(U)$. Let $\phi = \phi(b_1, \cdots, b_k)$ be any sentence of $\mathfrak{S}(U)$, where $b_1, \cdots, b_k$ are all the members of A occurring in it. Define $m_v(\phi) = \sum m(\phi(f(b_1), \cdots, f(b_k))) \cdot v([f])$ where the sum is taken over all $f \in F_0$ whose domain is $\{b_1, \cdots, b_k\}$. If no member of A occurs in ϕ then $m_v(\phi) = m(\phi)$. Note that if $\{b_1, \cdots, b_k\} \subseteq B$, where B is any finite subset of A, then $m_v(\phi) = \sum_{Df = B} m(\phi(f(b_1), \cdots, f(b_k))) v([f])$, this is so because if $Df = \{b_1, \cdots, b_k\}$ then $v([f]) = \sum v([g])$ where the sum is over all g's with $Dg = B$ and $g \mid Df = f$. For all $\phi, \vdash \phi(b_1, \cdots, b_k)$ implies $\vdash \phi(f(b_1), \cdots, f(b_k))$, therefore m_v is a measure on $\mathfrak{S}(U)$. By (ii) we have $m_v(\phi(b_1, \cdots, b_k)) = m_v(\phi(h(b_1), \cdots, h(b_k)))$ whenever $b_1, \cdots, b_k$ are all the members of A occurring in ϕ and h is a permutation of A.

Now consider a sentence $\phi(b_1, \cdots, b_k)$ of the form $\exists x \psi(b_1, \cdots, b_k, x)$, where $b_1, \cdots, b_k$ are all the members of A occurring in it. Assume that for all $b_1', \cdots, b_k' \in A$ we have $m(\phi(b_1', \cdots, b_k')) = \sup\{m(\bigvee_{i=1}^{n} \psi(b_1', \cdots, b_k', d_i) \mid d_1, \cdots, d_n \in U\}$. Given any $\varepsilon > 0$ there is a finite subset F_1 of F_0, all of whose members have $\{b_1, \cdots, b_k\}$ as a domain, such that:

(a) $\sum_{f \in F_1} m(\phi(f(b_1), \cdots, f(b_k))) \cdot v([f]) \geqq (1 - \varepsilon) m_v(\phi(b_1, \cdots, b_k))$

Put $B' = \bigcup_{f \in F_1} \mathfrak{a} f$. There is a finite subset U' of U such that:

(b) $m(\bigvee_{d \in U'} \psi(b_1', \cdots, b_k', d)) \geqq (1 - \varepsilon) \cdot m(\phi(b_1', \cdots, b_k'))$, for all $b_1', \cdots, b_k'$ in B'.

From (a) and (b) we get:

(c) $\sum_{f \in F_1} m(\bigvee_{d \in U'} \psi(f(b_1), \cdots, f(b_k), d)) \cdot v([f]) \geqq (1 - \varepsilon)^2 \cdot m_v(\phi(b_1, \cdots, b_k))$

Put $A' = U' \cap A$. Since A' is finite, (iii) implies that, for every $f \in F_0$, one gets

$$v(\{g \mid g \in F \,\&\, g \mid Df = f \,\&\, g(Dg - Df) \supseteq A'\}) = v([f]),$$

(if X is a set $g(X)$ is defined as $\{g(x) \mid x \in X\}$).

Consequently there is a finite subset F_2 of F_0, the domain of all whose members is $\{b_1, \cdots, b_k\} \cup A_1$, where A_1 is a finite subset of A, such that:

(d) $g(A_1) \supseteq A'$ for all $g \in F_2$, and ,

$$v(\bigcup_{g \in F_2 \,\&\, g \mid Df = f} [g]) \geqq (1 - \varepsilon) v([f])$$

for all $f \in F_1$.

Put $C' = U' \cap C$, then $U' = A' \cup C'$ and from (d) one gets:

(e)

$(1 - \varepsilon) \sum_{f \in F_1} m(\bigvee_{d \in U'} \psi(f(b_1), \cdots, f(b_k), d)) \cdot v([f]) =$

$(1 - \varepsilon) \sum_{f \in F_1} m(\bigvee_{d \in C'} \psi(f(b_1), \cdots, f(b_k), d) \vee \bigvee_{d \in A'} \psi(f(b_1), \cdots, f(b_k), d)) \cdot v([f]) \leqq$

$\sum_{f \in F_2} m(\bigvee_{d \in C'} \psi(f(b_1), \cdots, f(b_k), d) \vee \bigvee_{d \in A_1} \psi(f(b_1), \cdots, f(b_k), f(d))) \cdot v([f])$

Put $U_1 = C' \cup A_1$, then, by the definition of m_ν

$$m_\nu(\bigvee_{d\in U_1}\psi(b_1,\cdots,b_k,d)) \geqq \sum_{f\in F_2} m(\bigvee_{d\in C'}\psi(f(b_1),\cdots,f(b_k),d) \vee \bigvee_{d\in A_1}\psi((b_1),\cdots,f(b_k),f(d))\cdot\nu([f])$$

Hence it follows from (c) and (e) that

$$\text{(f)} \qquad m_\nu(\bigvee_{d\in U_1}\psi(b_1,\cdots,b_k,d)) \geqq (1-\varepsilon)^3 m_\nu(\phi(b_1,\cdots,b_k))$$

Hence:

$$\sup\{m_\nu(\bigvee_{i=1}^n\psi(b_1,\cdots,b_k,d_i))\,|\,d_1,\cdots,d_n\in U\} \geqq (1-\varepsilon)^3 m_\nu(\phi(b_1,\cdots,b_k))$$

Sending ε to 0 we get the inequality which implies that the supremum on the left side is equal to $m_\nu(\phi(b_1,\cdots,b_k))$. Consequently:

If m satisfies (3) so does m_ν.

Now let $\langle U,m\rangle$ be any measure-model for μ with $\bar{\bar{U}} \leqq \aleph_0$. Let m^* be the unique extension of m to $\mathfrak{S}(U)$ satisfying (3). $(m^*)_\nu$ satisfies (3) and coincides with μ on $D\mu$. The restriction of $(m^*)_\nu$ to $\mathfrak{S}_0(U)$ is m_ν, therefore $\langle U,m_\nu\rangle$ is the required measure-model. q.e.d.

Note that if all the individual constants occurring in the sentences of $D\mu$ are members of C_1, where C_1 is some subset of C, then one can construct a measure-model for μ which is symmetric in $U-C_1$, by replacing in the given construction C by C_1. In particular if $C_1 = \emptyset$ one gets a measure-model symmetric in U.

The theorem is not true for $\mathfrak{P}$ with an uncountable set of sentences where $\overline{\overline{D\mu}} > \aleph_0$. Consider the following example: Let $\mathfrak{P}$ be with no individual constants, let $P_\xi, \xi\in\Xi$ be its predicates, where every P_ξ is a one-place predicate and Ξ is uncountable. Let μ be a measure on $\mathfrak{P}$ such that $\mu(\exists x(P_\xi(x)\wedge P_\eta(x)) = 0$ whenever $\xi\neq\eta$, and $\mu(\exists xP_\xi(x)) = 1$ for all $\xi\in\Xi$. Let $\langle U,m\rangle$ be a measure-model for μ. If the model is symmetric in U we have $m(P_\xi(a)) = m(P_\xi(b))$ for all $a,b\in U$, hence we must have $m(P_\xi(a)) > 0$ for all $\xi\in\Xi$, $a\in U$. Hence in $\{m(P_\xi(a))\,|\,\xi\in\Xi\}$ there are infinitely many numbers $\geqq\varepsilon$, where ε is some fixed number >0. That is, $m(P_\xi(a))\geqq\varepsilon$ for all $\xi\in\Xi'$, where $\bar{\bar{\Xi}}' \geqq \aleph_0$. Also $m(P_\xi(a)\wedge P_\eta(a)) = 0$ for all $\xi\neq\eta$, therefore $m(\bigvee_{i=1}^n P_{\xi_i}(a)) = \sum_{i=1}^n m(P_{\xi_i}(a)) \geqq n\cdot\varepsilon$ for all $\xi_1,\cdots,\xi_n\in\Xi'$, contradicting the boundedness of m.

If $\mathfrak{P}$ has an equality, then the measure-model constructed in the proof of Theorem 4 is not with a strict equality. In fact, there are measures μ on first order calculi with equality, having countably many sentences, which do not have measure-models symmetric in $U-C$ with strict equality. For example, let $\mathfrak{P}$ have only one one-place predicate P and equality. Let μ be a measure on $\mathfrak{P}$ such that $\mu(\exists xP(x)) = 1$, $\mu(\forall x,y(P(x)\wedge P(y)\rightarrow x = y)) = 1$, and

$$\mu(\exists x_1,\cdots,x_k\forall y(\bigvee_{i=1}^k y = x_i)) = 0$$

for all k. If $\langle U,m\rangle$ is a measure-model for μ then $\bar{\bar{U}} \geqq \aleph_0$, and if it is symmetric in U then $m(P(a)) = \varepsilon > 0$ for all $a\in U$. If it is with strict equality then we must have $m(P(a) \wedge P(b)) = 0$ whenever $a \neq b$. Hence $m(\bigvee_{i=1}^{n} P(a_i)) = n \cdot \varepsilon$; taking n large enough we get the contradiction $m(\exists x P(x)) > 1$.

The problem of characterizing those theories (i.e. measures having the values 0 and 1) which possess a measure-model with strict equality, satisfying also the symmetry condition, seems to be difficult. As was pointed out by Ryll-Nardzewski, the theory of a linear dense ordering without first and last elements has such a model.

§5. **An example.** The following is a simple example of a measure-model suggested to the author by M. Rabin and D. Scott. Let $\mathfrak{P}$ be without individual constants, with or without equality. Let U be an infinite set. Define m on $\mathfrak{S}_0(U)$ by first putting $m(R(a_1, \cdots, a_n)) = q$ for all predicates R of $\mathfrak{P}$ and all $a_1, \cdots, a_n \in U$, excluding the case where $R(a_1, a_2)$ is $a_1 = a_2$. q is some fixed number, $0 < q < 1$. Next, for every conjunction $\bigwedge_{i=1}^{k} \phi_i \wedge \bigwedge_{i=k+1}^{n} \sim \phi_i$, where ϕ_i are atomic sentences which are not equalities and $\phi_i \neq \phi_j$ for $i \neq j$, define m to be $q^k(1-q)^{n-k}$. This defines m for all sentences of $\mathfrak{S}_0(U)$ which do not involve equalities. If $\mathfrak{P}$ has equality put $m(a = b) = 0$ whenever $a \neq b$, this determines m completely. $\langle U, m \rangle$ is a measure-model symmetric in U and the atomic sentences are "independent" in the sense that if $\phi, \psi \in \mathfrak{S}_0(U)$ and no atomic sentence, besides equalities, is a part of both ϕ and ψ then $m(\phi \wedge \psi) = m(\phi) \cdot m(\psi)$. Let m^* be the extension of m to $\mathfrak{S}(U)$ satisfying (3); we claim that whenever $\phi, \psi \in \mathfrak{S}(U)$ and no individual constant occurs both in ϕ and ψ then $m^*(\phi \wedge \psi) = m^*(\phi) \cdot m^*(\psi)$. This is proved for formulas in prenex normal form by induction on the sum of the numbers of alternating blocks of quantifiers appearing in the two sentences. If $\phi, \psi \in \mathfrak{S}_0(U)$ this is obvious. Assume, for the sake of simplicity, that $\phi = \exists x \phi_1(x)$ where $\phi_1(x)$ starts with a universal quantifier. Let A, B be the sets of constants occurring in ϕ and ψ, respectively. Consider $\bigvee_{a \in A'} \phi_1(a)$, where A' is some finite subset of U. Since $A \cap B = \emptyset$ there is a permutation h of U, satisfying $h(a) = a$ for all $a \in A$ and $B \cap h(A') = \emptyset$. The symmetry of m implies that m^* is also symmetric in U hence $m^*(\bigvee_{a \in A'} \phi_1(a)) = m^*(\bigvee_{a \in h(A')} \phi_1(a))$. It follows that $m^*(\phi) = \sup\{m^*(\bigvee_{i=1}^{n} \phi_1(a_i)) \mid a_1, \cdots, a_n \in U - B\}$. Consider now a sequence of sentences $\sigma_1, \sigma_2, \cdots$, if $\vdash \sigma_i \to \sigma_{i+1}$ and $\vdash \sigma_i \to \sigma$ for $i = 1, 2, \cdots$ and if $m^*(\sigma) = \lim_{i \to \infty} m^*(\sigma_i)$ then, for every τ, $m^*(\sigma \wedge \tau) = \lim_{i \to \infty} m^*(\sigma_i \wedge \tau)$; this becomes evident once we regard m^* as a countably additive measure on sets of models. Consequently we get

$$m^*(\psi \wedge \phi) = \sup\{m^*(\psi \wedge \bigvee_{i=1}^{n} \phi_1(a_i)) \mid a_1, \cdots, a_n \in U - B\}$$

hence, since $\bigvee_{i=1}^{n} (\phi_1 a_i)$ is logically equivalent to a formula with less alternating blocks, we get:

$$m^*(\psi \wedge \phi) = \sup\{m^*(\psi) \cdot m^*(\bigvee_{i=1}^{n} \phi_1(a_i)) \mid a_1, \cdots, a_n \in U - B\} = m^*(\psi) \cdot m^*(\phi)$$

The case where the first block of existential quantifiers has more than one quantifier is treated similarly. The case where the first quantifier is a universal one results from this by an easy calculation.

If $\mathfrak{S}$ is the set of sentences of $\mathfrak{P}$ we get, for every $\phi \in \mathfrak{S}$, $m^*(\phi) = m^*(\phi \wedge \phi) = (m^*(\phi))^2$, hence $m^*(\phi)$ is either 0 or 1. Therefore $\langle U, m \rangle$ is a measure-model of a complete theory. Note that this is some kind of zero-one law, however, the author does not see a way to deduce it directly as a special case of the zero-one law in probability theory.

To find out what complete theory is determined by $\langle U, m \rangle$ proceed as follows. Assume $\mathfrak{P}$ has an equality. Let $x_1, \cdots, x_n$ be distinct variables of $\mathfrak{P}$. By a complete diagram of $x_1, \cdots, x_n$ we mean a consistent conjunction of atomic formulas and negations of atomic formulas formed by using $x_1, \cdots, x_n$ and the predicates belonging to some finite set Σ (which does not include $=$) so that, for every k-place predicate R of Σ and every sequence $i_1, \cdots, i_k$, $1 \leqq i_j \leqq n$, (the i_j's not necessarily distinct), either $R(x_{i_1}, \cdots, x_{i_k})$ or $\sim R(x_{i_1}, \cdots, x_{i_k})$ is a conjunct. A complete diagram $\sigma_2(x_1, \cdots, x_{n+k})$ extends another complete diagram $\sigma_1(x_1, \cdots, x_n)$ if both use the same predicates and $\vdash \sigma_2 \rightarrow \sigma_1$. It is not difficult to show that $m^*(\exists x_1, \cdots, x_n \sigma_1(x_1, \cdots, x_n)) = 1$ and $m^*(\forall x_1, \cdots, x_n \exists y(\sigma_1(x_1, \cdots, x_n) \rightarrow \bigwedge_{i=1}^n y \neq x_i \wedge \sigma_2(x_1, \cdots, x_n, y))) = 1$ whenever σ_1 and σ_2 are complete diagrams and σ_2 extends σ_1. Let Φ be the set of all the sentences of these forms. Every member of Φ is a theorem in the complete theory. Moreover Φ is a set of axioms for this theory. To see it take the case where $\mathfrak{P}$ has finitely many predicates. In that case two countable models (in the usual sense) in which Φ holds are isomorphic. One proves this using the argument which proves the isomorphism of two countable dense linear orderings without extreme elements. The basic fact here is that given any isomorphism between two finite submodels and extending one of them by adding any extra element, one can add a suitable element to the other and extend the isomorphism to the bigger submodels. This follows directly from Φ.

A model for this theory can be described as "most general" in the sense that every possible finite model is realised there as a submodel, and for every finite submodel every possible finite extension of it is realised.

If $\mathfrak{P}$ has infinitely many predicates the same holds for all subtheories obtained by restriction to a finite family of predicates. However, there will always be two countable models for the theory which are not isomorphic.

If $\mathfrak{P}$ is without identity but has at least one k-place predicate, where $k \geqq 2$, then the situation is essentially the same. One gets a similar set, Φ, except that in the sentences the part $\bigwedge_{i=1}^n y \neq x_i$ is to be omitted. Let $\mathfrak{P}'$ be the calculus obtained by adding an equality to $\mathfrak{P}$. Assume, for simplicity, that R is a 2-place predicate of $\mathfrak{P}$. If $\sigma_1(x_1, \cdots, x_n)$ is a complete diagram in which R occurs, and $\sigma_2(x_1, \cdots, x_n, y)$ is a complete diagram extending σ_1, let $\sigma_2'(x_1, \cdots, x_n, y, y')$ be a complete diagram which extends σ_2 and has as conjuncts all the formulas $R(y', x_i)$, $1 \leqq i \leqq n$ and $\sim R(y', y)$. Φ implies $\forall x_1, \cdots, x_n \exists y, y'(\sigma_1(x_1, \cdots, x_n) \rightarrow$

$\sigma_2'(x_1,\cdots,x_n,y', y))$, but this, in $\mathfrak{P}'$, logically imples $\forall x_1,\cdots,x_n \exists y(\sigma_1(x_1,\cdots,x_n) \rightarrow \bigwedge_{i=1} y \neq x_i \wedge \sigma_2(x_1,\cdots,x_n,y))$. Since every complete diagram in which R does not occur is logically equivalent to a disjunction of complete diagrams in which R occurs, the same holds if R does not occur in σ_1. Therefore all the properties of the case in which we have equality carry over to this case.

If $\mathfrak{P}$ has no equality and all its predicates are one-place predicates, then the theory has as axioms all the sentences $\exists x\phi(x)$ where $\phi(x)$ is any consistent quantifier free formula. In case of a finite number of predicates this theory has finite models as well.

There is an easy way to eliminate quantifiers in the complete theory determined by $\langle U,m\rangle$. Let ϕ be a quantifier-free formula with free variables $x_1,\cdots,x_n,y$. It is logically equivalent to a formula of the form $\sigma_0 \vee \bigvee_{i=1}^{n}(\sigma_i \wedge \tau_i(y))$ where y does not occur in σ_i, $0 \leqq i \leqq n$, y occurs in every atomic formula which is a part of $\tau_i(y)$, $1 \leqq i \leqq n$, $\vdash \forall x_1,\cdots,x_k \sim (\sigma_i \wedge \sigma_j)$ whenever $i \neq j$, and $\tau_i(y)$ is not a tautology, $1 \leqq i \leqq n$. Now let σ_i', τ_i' be obtained by some fixed replacement of $x_1,\cdots,x_k$ by members of U. An easy calculation shows that

$$\inf\{m(\bigwedge_{j=1}^{k}(\sigma_0' \vee \bigvee_{i=1}^{n}\sigma_i' \wedge \tau_i'(a_j)) \,|\, a_1,\cdots,a_k \in U\} = m(\sigma_0')$$

Consequently $\forall y\phi(y)$ is equivalent in the theory to σ_0. The elimination is effective since, given ϕ, one can find σ_0 effectively.

Note that the theory one gets does not depend on the particular q with which one starts, provided only that $0 < q < 1$. The argument which was used to prove that $\langle U,m\rangle$ determines a complete theory can be used to prove that any measure-model satisfying the following requirements determines a complete theory.

(I) $\langle U,m\rangle$ is symmetric in U and U is infinite.

(II) If $\phi,\psi \in \mathfrak{S}_0(U)$ and no individual constant occurs in both then

$$m(\phi \wedge \psi) = m(\phi)\cdot m(\psi)$$

It is not difficult to show that the theory of a dense linear ordering without extreme elements has such a model.

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